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TRANSVERSE VIBRATIONS OF ELASTICALLY CONNECTED RECTANGULAR DOUBLE-MEMBRANE COMPOUND SYSTEM

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The free and forced vibrations of a system of two rectangular membranes attached together by a Winkler elastic layer are studied analytically. The motion of the system is described by two non-homogeneous partial differential equations. The solutions of the free vibrations are obtained by the Bernoulli–Fourier method. Solving the boundary value problem the natural frequencies and the mode shape functions are found. The initial-value problem is also solved. The free vibrations of the double-membrane system are realised by synchronous and asynchronous deflections. The forced vibrations of membranes subjected to arbitrarily distributed continuous loads are determined by using the classical method of the expansion in a series of the normal modes of vibrations. Discussing the vibrations caused by the harmonic exciting forces it is shown that the dynamic absorption phenomenon appears. Therefore, the double-membrane system can be used as a dynamic vibration absorber. As a numerical example the vibrations of the system consisting of two identical membranes subjected to harmonic uniform distributed load are treated in detail.

1. INTRODUCTION

The vibration analysis of a compound continuous systems with elastic constraints is of great theoretical and practical importance and has a wide application in aeronautics, cosmonautics, civil and mechanical engineering [1].

The compound continuous system considered consists of one-dimensional (string, ring, beam) or two-dimensional (membrane, plate) solids which are coupled by elastic layers. The simplest fundamental model of such a system is composed of two solids joined by a Winkler elastic layer (the elastically connected double-solid system).

In 1964 the transverse vibrations of an elastically connected double-beam system were considered by Seelig and Hoppmann II [2, 3]. This system has also been analyzed by Kessel [4], Kessel and Raske [5], Saito and Chonan [6, 7], Rao [8], Oniszczuk [9–15], Chonan [16, 17], Hamada *et al*. [18, 19], Yankelevsky [20], Kukla and Skalmierski [21]. The vibration problem concerning a similar double-string system has been solved by Oniszczuk [22]. The in-plane free

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vibrations of an elastically connected concentric two-ring systems have been investigated by Stead *et al*. [23], Kirkhope [24], Kunukkasseril and Reddy [25], and Rao [26, 27]. The transverse vibrations of circular and rectangular double-membrane systems have been discussed by Oniszczuk [28–31]. The very important and difficult problem of the transverse vibrations of rectangular and circular plates joined by an elastic layer has been studied by Kunukkasseril and Radhakrishnan [32], Kunukkasseril and Swamidas [33, 34], Chonan [35, 36], and Oniszczuk [37, 38].

In this paper the transverse vibrations of two rectangular membranes connected by a Winkler elastic layer are considered and the complete analytical solutions of free and forced vibrations are presented.

2. FORMULATION OF THE PROBLEM

The mechanical model of the vibrating system under consideration is composed of two parallel rectangular membranes connected by a massless, linear, elastic layer of Winkler type (see Figure 1). It is assumed that the membranes are thin, homogeneous and perfectly elastic and they have constant thickness. The membranes are uniformly tight by suitable constant tensions applied at the boundaries. The membranes are subjected to arbitrarily distributed continuous loads. The small vibrations of the system with no damping are considered.

The governing differential equations of the transverse vibrations of a double-membrane system have the following form [28–31]:

$$m_1\ddot{w}_1 - N_1\Delta w_1 + k(w_1 - w_2) = f_1, \qquad m_2\ddot{w}_2 - N_2\Delta w_2 + k(w_2 - w_1) = f_2, \quad (1)$$

where $w_i = w_i(x, y, t)$ is the transverse membrane displacement; $f_i = f_i(x, y, t)$ is the exciting distributed load; x, y, t are the space co-ordinates and the time; k is the stiffness modulus of a Winkler elastic layer; a, b, h_i are the membrane

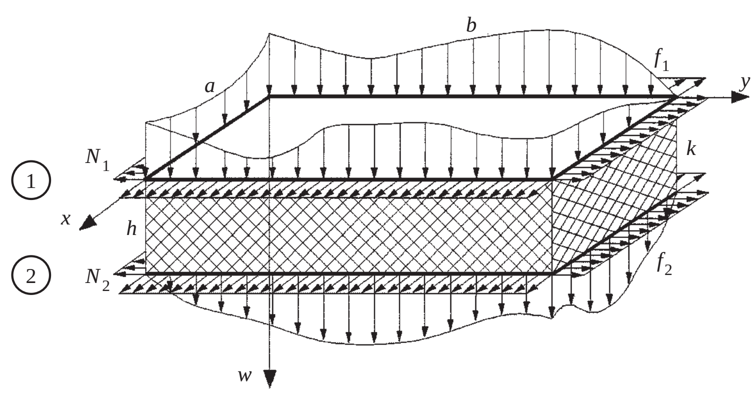

Figure 1. The physical model of an elastically connected rectangular double-membrane compound system.

dimensions; ρ_i is the mass density; N_i is the uniform constant tension per unit length;

$$m_i = \rho_i h_i, \qquad \dot{w}_i = \frac{\partial w_i}{\partial t}, \qquad \Delta w_i = \frac{\partial^2 w_i}{\partial x^2} + \frac{\partial^2 w_i}{\partial y^2}, \qquad i = 1, 2.$$

The boundary and initial conditions may be written as follows

$$w_i(0, y, t) = w_i(a, y, t) = w_i(x, 0, t) = w_i(x, b, t) = 0, \tag{2}$$

$$w_i(x, y, 0) = w_{i0}(x, y), \qquad \dot{w}_i|_{(x,y,0)} = v_{i0}(x, y), \qquad i = 1, 2. \tag{3}$$

3. FREE VIBRATIONS

The free vibrations of an elastically connected double-membrane system are described by two homogeneous differential equations:

$$m_1\ddot{w}_1 - N_1\Delta w_1 + k(w_1 - w_2) = 0, \qquad m_2\ddot{w}_2 - N_2\Delta w_2 + k(w_2 - w_1) = 0. \tag{4}$$

Using the Bernoulli–Fourier method (separation of variables) the general solutions of equations (4) are taken in the form:

$$w_i(x, y, t) = W_i(x, y)T(t), \qquad i = 1, 2, \tag{5}$$

$$T(t) = C \sin(\omega t) + D \cos(\omega t), \tag{6}$$

where ω is the natural frequency of the system. Substituting solutions (5) into equations (4) results in the following equations:

$$N_1\Delta W_1 + (m_1\omega^2 - k)W_1 + kW_2 = 0, \qquad N_2\Delta W_2 + (m_2\omega^2 - k)W_2 + kW_1 = 0. \tag{7}$$

Now by eliminating the function W_2 one gets the equation

$$\Delta^2 W_1 + [(m_1\omega^2 - k)N_1^{-1} + (m_2\omega^2 - k)N_2^{-1}]\Delta W_1$$
$$+\omega^2(N_1N_2)^{-1}[m_1m_2\omega^2 - k(m_1 + m_2)]W_1 = 0$$

or

$$(\Delta + k_1^2)(\Delta + k_2^2)W_1 = 0, \tag{8}$$

where

$$k_{1,2}^2 = 0{\cdot}5\{[(m_1\omega^2 - k)N_1^{-1} + (m_2\omega^2 - k)N_2^{-1}] \pm ([(m_1\omega^2 - k)N_1^{-1}$$
$$+(m_2\omega^2 - k)N_2^{-1}]^2 - 4\omega^2(N_1N_2)^{-1}[m_1m_2\omega^2 - k(m_1 + m_2)])^{1/2}\}. \tag{9}$$

The coefficients k_1^2 and k_2^2 are both positive when

$$\omega^2 > \omega_0^2 = k(m_1^{-1} + m_2^{-1}). \tag{10}$$

The harmonic type of free vibrations is assured by the condition (10). The solution of equation (8) has the form

$$W_1(x, y) = X(x)Y(y). \tag{11}$$

Substituting the expression (11) into an equation of type (8)

$$(\Delta + k_i^2)W_1 = 0, \qquad i = 1, 2, \tag{12}$$

gives the relation

$$X''Y + XY'' + k_i^2 XY = 0, \tag{13}$$

where

$$X' = \frac{\mathrm{d}X}{\mathrm{d}x}, \qquad Y' = \frac{\mathrm{d}Y}{\mathrm{d}y}.$$

Separating of the variables in equation (13) gives two independent ordinary differential equations

$$X'' + a_i^2 X = 0, \qquad Y'' + b_i^2 Y = 0, \tag{14}$$

where

$$k_i^2 = a_i^2 + b_i^2, \qquad i = 1, 2. \tag{15}$$

Solving the equations (14) yields the expressions [28]

$$X_i(x) = A_{1i}\sin(a_i x) + A_{2i}\cos(a_i x), \qquad Y_i(y) = B_{1i}\sin(b_i y) + B_{2i}\cos(b_i y). \tag{16}$$

Then the general mode shape function W_1 is found to be

$$W_1(x, y) = \sum_{i=1}^{2} W_{1i}(x, y) = \sum_{i=1}^{2} X_i(x)Y_i(y)$$

$$= \sum_{i=1}^{2} [A_{1i}\sin(a_i x) + A_{2i}\cos(a_i x)][B_{1i}\sin(b_i y) + B_{2i}\cos(b_i y)]. \tag{17}$$

Using the first equation of the system (7) one can now determine the general mode shape function W_2 in the following form:

$$W_2(x, y) = \sum_{i=1}^{2} W_{2i}(x, y) = \sum_{i=1}^{2} c_i W_{1i}(x, y) = \sum_{i=1}^{2} c_i X_i(x)Y_i(y)$$

$$= \sum_{i=1}^{2} [A_{1i}\sin(a_i x) + A_{2i}\cos(a_i x)][B_{1i}\sin(b_i y) + B_{2i}\cos(b_i y)]c_i, \tag{18}$$

where

$$c_i = (N_1 k_i^2 + k - m_1\omega^2)k^{-1} = k(N_2 k_i^2 + k - m_2\omega^2)^{-1}, \qquad i = 1, 2, \tag{19}$$

$$c_{1,2} = 0{\cdot}5k^{-1}N_1\{[(m_2\omega^2 - k)N_2^{-1} - (m_1\omega^2 - k)N_1^{-1}] \pm ([(m_2\omega^2 - k)N_2^{-1}$$
$$-(m_1\omega^2 - k)N_1^{-1}]^2 + 4k^2(N_1N_2)^{-1})^{1/2}\}, \qquad c_1 > 0, \qquad c_2 < 0.$$

The unknown constants A_{1i}, A_{2i}, B_{1i}, B_{2i} are found by solving the boundary value problem. Substituting the shape functions W_1 and W_2 into the boundary conditions (2) gives a set of eight homogeneous equations for the unknown constants. Solving it shows that $B_{1i} = B_{2i} = 0$ and the following characteristic equations are received

$$\sin(a_i a) = 0, \qquad \sin(b_i b) = 0, \qquad i = 1, 2. \tag{20}$$

From these equations the unknown coefficients a_i, b_i and k_i can be calculated

$$a_i = a_{im} = a_m = m\pi a^{-1}, \qquad b_i = b_{in} = b_n = n\pi b^{-1}, \qquad i = 1, 2, \tag{21}$$

$$k_i^2 = k_{imn}^2 = k_{mn}^2 = a_m^2 + b_n^2 = \pi^2[(a^{-1}m)^2 + (b^{-1}n)^2], \qquad m, n = 1, 2, 3, \ldots \tag{22}$$

Transforming properly the expression (9) gives the following frequency equation [28]:

$$\omega^4 - [(N_1 k_{mn}^2 + k)m_1^{-1} + (N_2 k_{mn}^2 + k)m_2^{-1}]\omega^2 + k_{mn}^2 (m_1 m_2)^{-1}[N_1 N_2 k_{mn}^2 + k(N_1 + N_2)] = 0. \tag{23}$$

The natural frequencies of the double-membrane system are determined from the formula:

$$\omega_{1,2mn}^2 = 0{\cdot}5\{[(N_1 k_{mn}^2 + k)m_1^{-1} + (N_2 k_{mn}^2 + k)m_2^{-1}] \mp ([(N_1 k_{mn}^2 + k)m_1^{-1} + (N_2 k_{mn}^2 + k)m_2^{-1}]^2 - 4k_{mn}^2 (m_1 m_2)^{-1}[N_1 N_2 k_{mn}^2 + k(N_1 + N_2)])^{1/2}\},$$

$$\omega_{1mn} < \omega_{2mn}. \tag{24}$$

One can now formulate the time functions (6) and the mode shapes of free vibrations (17), (18) corresponding to the natural frequencies ω_{imn}

$$T_{imn}(t) = C_{imn}\sin(\omega_{imn}t) + D_{imn}\cos(\omega_{imn}t), \tag{25}$$

$$W_{1imn}(x, y) = W_{mn}(x, y) = X_m(x)Y_n(y) = \sin(a_m x)\sin(b_n y),$$

$$W_{2imn}(x, y) = c_{imn}W_{mn}(x, y) = c_{imn}X_m(x)Y_n(y) = c_{imn}\sin(a_m x)\sin(b_n y), \tag{26}$$

where

$$c_{imn} = (N_1 k_{mn}^2 + k - m_1\omega_{imn}^2)k^{-1} = k(N_2 k_{mn}^2 + k - m_2\omega_{imn}^2)^{-1}, \tag{27}$$

$$c_{1,2mn} = 0{\cdot}5k^{-1}m_1\{[(N_1 k_{mn}^2 + k)m_1^{-1} - (N_2 k_{mn}^2 + k)m_2^{-1}] \pm ([(N_1 k_{mn}^2 + k)m_1^{-1} - (N_2 k_{mn}^2 + k)m_2^{-1}]^2 + 4k^2(m_1 m_2)^{-1})^{1/2}\},$$

$$W_{mn}(x, y) = \sin(a_m x)\sin(b_n y), \qquad X_m(x) = \sin(a_m x), \qquad Y_n(y) = \sin(b_n y),$$

$$c_{1mn} > 0, \qquad c_{2mn} < 0, \qquad c_{1mn}c_{2mn} = -m_1 m_2^{-1}, \qquad i = 1, 2, \qquad m, n = 1, 2, 3, \ldots$$

Finally the general solutions of the free vibrations of an elastically connected double-membrane system under consideration may be written in the following form:

$$w_1(x, y, t) = \sum_{(i,m,n)} W_{1imn}(x, y) T_{imn}(t) = \sum_{m,n=1}^{\infty} W_{mn}(x, y) \sum_{i=1}^{2} T_{imn}(t)$$

$$= \sum_{m,n=1}^{\infty} \sin(a_m x) \sin(b_n y) \sum_{i=1}^{2} [C_{imn} \sin(\omega_{imn} t) + D_{imn} \cos(\omega_{imn} t)],$$

$$w_2(x, y, t) = \sum_{(i,m,n)} W_{2imn}(x, y) T_{imn}(t) = \sum_{m,n=1}^{\infty} W_{mn}(x, y) \sum_{i=1}^{2} c_{imn} T_{imn}(t)$$

$$= \sum_{m,n=1}^{\infty} \sin(a_m x) \sin(b_n y) \sum_{i=1}^{2} [C_{imn} \sin(\omega_{imn} t) + D_{imn} \cos(\omega_{imn} t)] c_{imn}. \tag{29}$$

The free vibrations of membranes are realised in the form of synchronous ($c_{1mn} > 0, \omega_{1mn}$) and asynchronous ($c_{2mn} < 0, \omega_{2mn}$) displacements. The solution of the initial-value problem requires the knowing of the orthogonality condition of normal modes of vibrations. This condition is built using the equations (7) rewritten in the following form:

$$N_1 \Delta W_{1imn} + (m_1 \omega_{imn}^2 - k) W_{1imn} + k W_{2imn} = 0,$$

$$N_2 \Delta W_{2imn} + (m_2 \omega_{imn}^2 - k) W_{2imn} + k W_{1imn} = 0.$$

With the expressions (26) and (27) we can transform the above in the equation as for a single membrane

$$\Delta W_{mn} + k_{mn}^2 W_{mn} = 0. \tag{30}$$

Then the orthogonality condition of mode shape functions has the known classical form:

$$\int_0^a \int_0^b W_{kl} W_{mn} \, \mathrm{d}x \, \mathrm{d}y = \int_0^a \sin(a_k x) \sin(a_m x) \, \mathrm{d}x \int_0^b \sin(b_l y) \sin(b_n y) \, \mathrm{d}y$$

$$= \begin{cases} 0, & k \neq m \quad \text{or} \quad l \neq n, \\ a_{mn}^2, & k = m \quad \text{and} \quad l = n, \end{cases} \tag{31}$$

where

$$a_{mn}^2 = \int_0^a \int_0^b W_{mn}^2 \, \mathrm{d}x \, \mathrm{d}y = \int_0^a \sin^2(a_m x) \, \mathrm{d}x \int_0^b \sin^2(b_n y) \, \mathrm{d}y = 0{\cdot}25ab.$$

Substituting the solutions (29) into the initial conditions (3) gives the relations

$$w_{10} = \sum_{(m,n)} W_{mn} \sum_{i=1}^{2} D_{imn}, \qquad v_{10} = \sum_{(m,n)} W_{mn} \sum_{i=1}^{2} \omega_{imn} C_{imn},$$

$$w_{20} = \sum_{(m,n)} W_{mn} \sum_{i=1}^{2} c_{imn} D_{imn}, \qquad v_{20} = \sum_{(m,n)} W_{mn} \sum_{i=1}^{2} c_{imn} \omega_{imn} C_{imn}.$$

Multiplying these relations by the eigenfunction W_{kl} then integrating them over the membrane surface and using the orthogonality condition (31) produces

$$\int_0^a \int_0^b w_{10} W_{mn}\, \mathrm{d}x\, \mathrm{d}y = a_{mn}^2 \sum_{i=1}^{2} D_{imn}, \qquad \int_0^a \int_0^b v_{10} W_{mn}\, \mathrm{d}x\, \mathrm{d}y = a_{mn}^2 \sum_{i=1}^{2} \omega_{imn} C_{imn},$$

$$\int_0^a \int_0^b w_{20} W_{mn}\, \mathrm{d}x\, \mathrm{d}y = a_{mn}^2 \sum_{i=1}^{2} c_{imn} D_{imn},$$

$$\int_0^a \int_0^b v_{20} W_{mn}\, \mathrm{d}x\, \mathrm{d}y = a_{mn}^2 \sum_{i=1}^{2} c_{imn} \omega_{imn} C_{imn},$$

from where one can obtain the following formulas making it possible to calculate the unknown constants:

$$C_{1mn} = (\omega_{1mn} z_{1mn})^{-1} \int_0^a \int_0^b (c_{2mn} v_{10} - v_{20}) \sin(a_m x) \sin(b_n y)\, \mathrm{d}x\, \mathrm{d}y,$$

$$C_{2mn} = (\omega_{2mn} z_{2mn})^{-1} \int_0^a \int_0^b (c_{1mn} v_{10} - v_{20}) \sin(a_m x) \sin(b_n y)\, \mathrm{d}x\, \mathrm{d}y,$$

$$D_{1mn} = z_{1mn}^{-1} \int_0^a \int_0^b (c_{2mn} w_{10} - w_{20}) \sin(a_m x) \sin(b_n y)\, \mathrm{d}x\, \mathrm{d}y,$$

$$D_{2mn} = z_{2mn}^{-1} \int_0^a \int_0^b (c_{1mn} w_{10} - w_{20}) \sin(a_m x) \sin(b_n y)\, \mathrm{d}x\, \mathrm{d}y, \tag{32}$$

where

$$z_{1mn} = -z_{2mn} = a_{mn}^2 (c_{2mn} - c_{1mn}) = 0{\cdot}25ab(c_{2mn} - c_{1mn}).$$

4. FORCED VIBRATIONS

The forced vibrations of two membranes subjected to arbitrarily distributed continuous loads are determined by using the classical method of the expansion in a series of the normal modes of vibrations.

The particular solutions of non-homogeneous differential equations (1) representing the forced vibrations of double-membrane system are assumed in the form:

$$w_1(x, y, t) = \sum_{(i,m,n)} W_{1imn}(x, y) S_{imn}(t) = \sum_{m,n=1}^{\infty} W_{mn}(x, y) \sum_{i=1}^{2} S_{imn}(t),$$

$$w_2(x, y, t) = \sum_{(i,m,n)} W_{2imn}(x, y) S_{imn}(t) = \sum_{m,n=1}^{\infty} W_{mn}(x, y) \sum_{i=1}^{2} c_{imn} S_{imn}(t), \qquad (33)$$

where $S_{imn}(t)$ are the unknown time functions corresponding to the natural frequencies ω_{imn}. Substituting the solutions (33) into the governing equations (1) gives

$$\sum_{(m,n)} \left\{ W_{mn} \sum_{i=1}^{2} [m_1 \ddot{S}_{imn} + k(1 - c_{imn}) S_{imn}] - N_1 \Delta W_{mn} \sum_{i=1}^{2} S_{imn} \right\} = f_1,$$

$$\sum_{(m,n)} \left\{ W_{mn} \sum_{i=1}^{2} c_{imn} [m_2 \ddot{S}_{imn} + k(1 - c_{imn}^{-1}) S_{imn}] - N_2 \Delta W_{mn} \sum_{i=1}^{2} c_{imn} S_{imn} \right\} = f_2.$$

Taking equations (27) and (30) into consideration gives

$$m_1 \sum_{(m,n)} W_{mn} \sum_{i=1}^{2} (\ddot{S}_{1mn} + \omega_{imn}^2 S_{imn}) = f_1,$$

$$m_2 \sum_{(m,n)} W_{mn} \sum_{i=1}^{2} (\ddot{S}_{imn} + \omega_{imn}^2 S_{imn}) c_{imn} = f_2.$$

Multiplying both sides of the above equations by the eigenfunction W_{kl} then integrating over the membrane surface and using the orthogonality condition (31) gives a set of equations from which the differential equations for unknown time functions are found

$$\ddot{S}_{imn} + \omega_{imn}^2 S_{imn} = K_{imn}(t), \qquad i = 1, 2, \qquad (34)$$

where

$$K_{1mn}(t) = z_{1mn}^{-1} \int_0^a \int_0^b (c_{2mn} m_1^{-1} f_1 - m_2^{-1} f_2) W_{mn} \, \mathrm{d}x \, \mathrm{d}y,$$

$$K_{2mn}(t) = z_{2mn}^{-1} \int_0^a \int_0^b (c_{1mn} m_1^{-1} f_1 - m_2^{-1} f_2) W_{mn} \, \mathrm{d}x \, \mathrm{d}y,$$

$$z_{1mn} = -z_{2mn} = 0{\cdot}25ab(c_{2mn} - c_{1mn}).$$

Their solutions satisfying the zero initial conditions are as follows

$$S_{imn}(t) = \omega_{imn}^{-1} \int_0^t K_{imn}(s) \sin [\omega_{imn}(t - s)] \, \mathrm{d}s, \qquad i = 1, 2. \tag{35}$$

Finally the expressions describing the forced vibrations of an elastically connected double-membrane system have the following form:

$$w_1(x, y, t) = \sum_{m,n=1}^{\infty} \sin (a_m x) \sin (b_n y) \sum_{i=1}^{2} \omega_{imn}^{-1} \int_0^t K_{imn}(s) \sin [\omega_{imn}(t - s)] \, \mathrm{d}s,$$

$$w_2(x, y, t) = \sum_{m,n=1}^{\infty} \sin (a_m x) \sin (b_n y) \sum_{i=1}^{2} c_{imn}\omega_{imn}^{-1} \int_0^t K_{imn}(s) \sin [\omega_{imn}(t - s)] \, \mathrm{d}s. \tag{36}$$

As an example the interesting particular case of load is now considered. The calculation is carried out for harmonic distributed load applied only to the first membrane

$$f_1(x, y, t) = f(x, y) \sin (pt), \qquad f_2(x, y, t) = 0,$$

where p is the forcing frequency.

The steady state forced vibrations of membranes are obtained in the following form:

$$w_1(x, y, t) = \sin (pt) \sum_{m,n=1}^{\infty} A_{1mn} \sin (a_m x) \sin (b_n y),$$

$$w_2(x, y, t) = \sin (pt) \sum_{m,n=1}^{\infty} A_{2mn} \sin (a_m x) \sin (b_n y), \tag{37}$$

where

$$A_{1mn} = 4F_{mn} M_1^{-1}(\omega_{22mn}^2 - p^2)[(\omega_{1mn}^2 - p^2)(\omega_{2mn}^2 - p^2)]^{-1},$$
$$A_{2mn} = 4F_{mn} K^{-1} \omega_{12}^4 [(\omega_{1mn}^2 - p^2)(\omega_{2mn}^2 - p^2)]^{-1}, \tag{38}$$

$$F_{mn} = \int_0^a \int_0^b f(x, y) \sin (a_m x) \sin (b_n y) \, \mathrm{d}x \, \mathrm{d}y, \qquad K = abk,$$

$$M_i = abm_i = abh_i\rho_i, \qquad i = 1, 2, \qquad \omega_{12}^4 = k^2(m_1 m_2)^{-1} = K^2(M_1 M_2)^{-1},$$
$$\omega_{22mn}^2 = (N_2 k_{mn}^2 + k) m_2^{-1} = (abN_2 k_{mn}^2 + K) M_2^{-1}.$$

The analysis of amplitudes (38) leads to the following conditions:

(a) condition of resonance

$$p = \omega_{imn}, \qquad i = 1, 2, \qquad m, n = 1, 2, 3, \ldots,$$

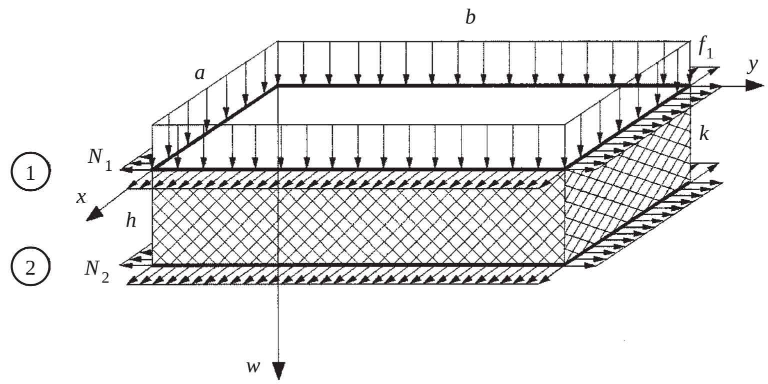

Figure 2. An elastically connected double-membrane system subjected to harmonic uniform distributed load.

(b) condition of dynamic vibration absorption

$$A_{1mn} = 0, \qquad A_{2mn} = -4F_{mn}K^{-1},$$

$$p^2 = p_{mn}^2 = \omega_{22mn}^2 = (abN_2k_{mn}^2 + K)M_2^{-1}.$$

It is proved that the second membrane acts like a dynamic vibration absorber in relation to the first one (main body). Suitable choice of elastic layer stiffness modulus (k), tension force (N_2) and second membrane mass (M_2) causes the appearance of dynamic absorption phenomenon. The dynamic absorption eliminates any selected harmonic component of first membrane vibrations. In the compound continuous system the dynamic absorber reduces the forced vibrations of the main body but never liquidates them absolutely [1]. The dynamic absorption phenomenon is of great practical importance.

5. NUMERICAL EXAMPLE

The system of two identical rectangular membranes are considered. The first membrane is subjected to harmonic uniform load which is distributed continuously on its whole surface (see Figure 2):

$$f_1(x, y, t) = f \sin(pt), \qquad f_2(x, y, t) = 0.$$

The following values of the parameters are used in the numerical calculations:

$$a = 1\text{ m}, \quad b = 2\text{ m}, \quad h = h_i = 1 \times 10^{-3}\text{ m}, \quad i = 1, 2, \quad k = 2 \times 10^2\text{ N m}^{-3},$$

$$M = m_i = \rho h = 2 \times 10^{-2}\text{ kg m}^{-2}, \quad N = N_i = 50\text{ N m}^{-1}, \quad \rho = \rho_i = 20\text{ kg m}^{-3}.$$

The initial conditions are assumed as follows:

$$w_{10}(x, y) = w_0 \sin(a^{-1}\pi x)\sin(b^{-1}\pi y), \qquad w_{20} = v_{10} = v_{20} = 0.$$

The general solutions of free vibrations (29) have the form:

$$w_1(x, y, t) = \sum_{m,n=1}^{\infty} \sin(a_m x)\sin(b_n y)\sum_{i=1}^{2}[C_{imn}\sin(\omega_{imn}t) + D_{imn}\cos(\omega_{imn}t)],$$

$$w_2(x, y, t) = \sum_{m,n=1}^{\infty} \sin(a_m x)\sin(b_n y)\sum_{i=1}^{2}[C_{imn}\sin(\omega_{imn}t) + D_{imn}\cos(\omega_{imn}t)]c_{imn},$$

where the natural frequencies and the mode shape coefficients are received from the expressions (21), (22), (24) and (27)

$$a_m = a^{-1}m\pi, \quad b_n = b^{-1}n\pi, \quad k_{mn}^2 = \pi^2[(a^{-1}m)^2 + (b^{-1}n)^2], \quad c_{1mn} = -c_{2mn} = 1,$$

$$\omega_{1mn}^2 = M^{-1}Nk_{mn}^2, \qquad \omega_{2mn}^2 = \omega_{1mn}^2 + \omega_0^2, \qquad \omega_0^2 = 2kM^{-1}, \qquad m, n = 1, 2, 3, \ldots$$

The results of the calculations of the natural frequencies are presented in Table 1. The mode shapes of vibrations corresponding to the first four pairs of the natural frequencies are shown in Figure 3. The natural mode shapes of vibrations are described by the expressions

$$W_{1imn} = W_{mn}, \qquad W_{2imn} = c_{imn}W_{mn}, \qquad W_{mn} = \sin(m\pi x)\sin(0{\cdot}5n\pi y),$$

$$c_{1mn} = -c_{2mn} = 1, \qquad i = 1, 2.$$

The double-membrane system executes two kinds of vibrations: in-phase (synchronous) vibrations ($c_{1mn} > 0$) with lower frequencies ω_{1mn} ($\omega_{1mn} < \omega_{2mn}$) and out-of-phase (asynchronous) vibrations ($c_{2mn} < 0$) with higher frequencies ω_{2mn}. The deflection form of membrane surface is identical for any pair of natural frequencies ω_{imn}. The synchronous vibrations are performed by both membranes with equal amplitudes ($c_{1mn} = 1$) then the elastic layer is not deformed on the

TABLE 1

Natural frequencies of double-membrane system $\omega_{imn}(\mathrm{s}^{-1})$

m	n / ω_{imn}	1 (ω_{1m1}, ω_{2m1})	2 (ω_{1m2}, ω_{2m2})	3 (ω_{1m3}, ω_{2m3})	4 (ω_{1m4}, ω_{2m4})	5 (ω_{1m5}, ω_{2m5})	6 (ω_{1m6}, ω_{2m6})
1	ω_{11n}	175·6	222·1	283·2	351·2	422·9	496·7
	ω_{21n}	225·5	263·3	316·5	378·6	446·0	516·5
2	ω_{12n}	323·8	351·2	392·7	444·3	502·9	566·4
	ω_{22n}	353·4	378·6	417·4	466·3	522·4	583·8
3	ω_{13n}	477·7	496·7	526·9	566·4	613·4	666·4
	ω_{23n}	498·2	516·5	545·5	583·8	629·5	681·3
4	ω_{14n}	633·2	647·7	671·0	702·5	740·9	785·4
	ω_{24n}	648·8	662·9	685·8	716·6	754·3	798·0
5	ω_{15n}	789·2	800·9	820·0	845·9	878·1	915·9
	ω_{25n}	801·9	813·3	832·1	857·6	889·4	926·8
6	ω_{16n}	945·7	955·5	971·5	993·5	1021·0	1053·7
	ω_{26n}	956·3	965·9	981·7	1003·5	1030·8	1063·2

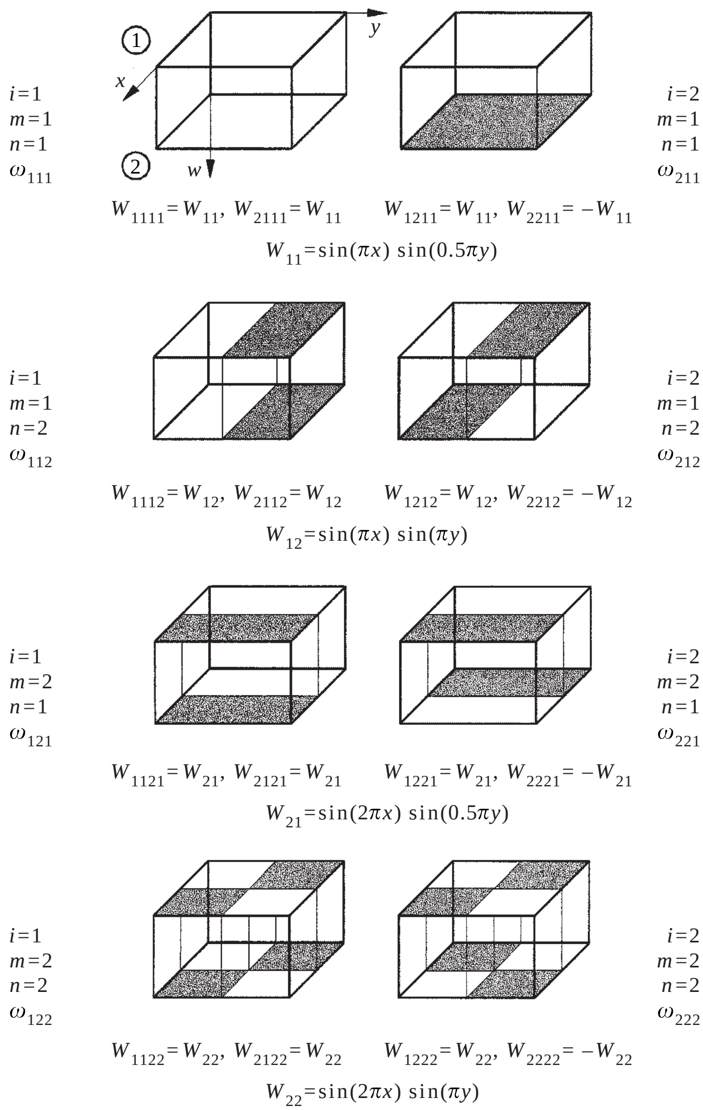

Figure 3. The mode shapes of vibrations of a rectangular double-membrane system corresponding to the first four pairs of the natural frequencies.

transverse direction. In this case the double-membrane system oscillates as a single membrane with the same natural frequencies. The natural frequencies of the asynchronous vibrations are identical as for the single membrane vibrating on the elastic layer of stiffness modulus $2k$.

Solving the initial-value problem the free vibrations of membranes are found in the final form

$$w_1(x, y, t) = 0{\cdot}5w_0 \sin(\pi x) \sin(0{\cdot}5\pi y)[\cos(\omega_{111} t) + \cos(\omega_{211} t)],$$

$$w_2(x, y, t) = 0{\cdot}5w_0 \sin(\pi x) \sin(0{\cdot}5\pi y)[\cos(\omega_{111} t) - \cos(\omega_{211} t)].$$

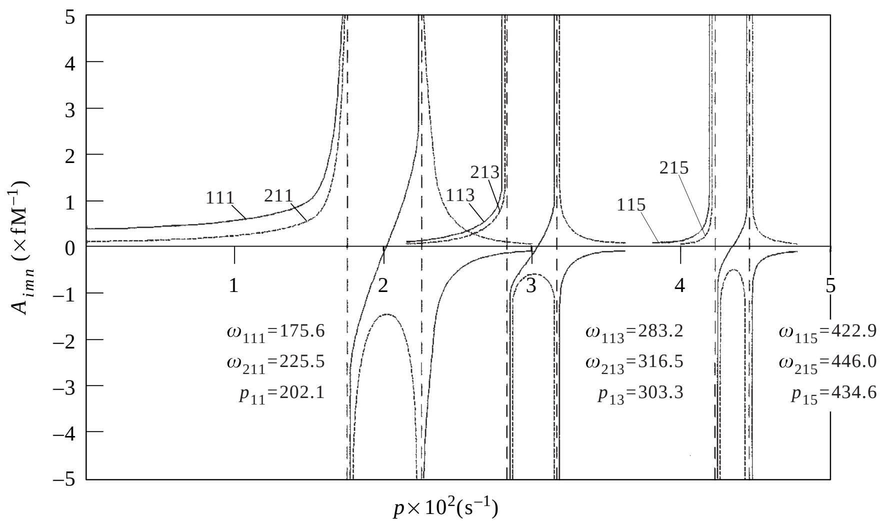

Figure 4. The resonance curves of a rectangular double-membrane system subjected to harmonic uniform distributed load.

The assumed initial conditions cause the membrane vibrations with the first pair frequencies ω_{111} and ω_{211}. The membranes execute the synchronous vibrations with lower frequency $\omega_{111} = 175{\cdot}6(\mathrm{s}^{-1})$ (and the equal amplitudes) and asynchronous vibrations with higher frequency $\omega_{211} = 225{\cdot}5(\mathrm{s}^{-1})$ (see Figure 3).

The steady-state forced vibrations of the membrane system are determined from the expressions (38) and (39) for the case of harmonic uniform distributed load

$$w_1(x, y, t) = \sin(pt) \sum_{(m,n)} A_{1mn} \sin(a_m x) \sin(b_n y),$$

$$w_2(x, y, t) = \sin(pt) \sum_{(m,n)} A_{2mn} \sin(a_m x) \sin(b_n y),$$

where

$$A_{1mn} = 8fM^{-1}(mn\pi^2)^{-1}(\omega_{1mn}^2 + \omega_{2mn}^2 - 2p^2)[(\omega_{1mn}^2 - p^2)(\omega_{2mn}^2 - p^2)]^{-1},$$

$$A_{2mn} = 8fM^{-1}(mn\pi^2)^{-1}\omega_0^2[(\omega_{1mn}^2 - p^2)(\omega_{2mn}^2 - p^2)]^{-1}, \qquad m, n = 1, 3, 5, \ldots$$

The forced vibrations are expressed only by the symmetric mode shapes because of the symmetry of the applied load. The first three resonance curves of the forced vibrations of the double-membrane system are presented in Figure 4. The full lines 111, 113, 115 describe the amplitudes of synchronous vibration components A_{111}, A_{113}, A_{115} and the broken lines 211, 213, 215 represent the amplitudes of asynchronous vibration components A_{211}, A_{213}, A_{215}. The quantities p_{11}, p_{13}, p_{15} are the exciting frequencies at which the dynamic vibration absorption occurs. These frequencies are calculated from the condition

$$p^2 = p_{mn}^2 = \omega_{22mn}^2 = (Nk_{mn}^2 + k)M^{-1} = 0{\cdot}5(\omega_{1mn}^2 + \omega_{2mn}^2),$$

which leads to the following membrane amplitudes:

$$A_{1mn} = 0, \qquad A_{2mn} = -32fM^{-1}(mn\pi^2\omega_0^2)^{-1} = -16fk^{-1}(mn\pi^2)^{-1},$$

$$m, n = 1, 3, 5, \ldots$$

6. CONCLUSIONS

This work deals with the transverse vibrations of an elastically connected rectangular double-membrane system. The free vibrations are determined by using the Bernoulli–Fourier method. It is shown that the membranes perform both synchronous and asynchronous motions. The forced vibrations caused by arbitrarily distributed continuous loads are found by the method of the expansion in a series of the mode shape functions. In the case of action of the harmonic forces, dynamic vibration absorption occurs and the double-membrane system can be used as a dynamic vibration absorber. This phenomenon is of great practical importance.

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